Li *Boundary Value Problems* (2016) 2016:192
DOI 10.1186/s13661-016-0701-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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A free boundary problem for a ratio-dependent diffusion predator-prey system

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Abstract

A ratio-dependent diffusion predator-prey system with free boundary is investigated to understand the impact of free boundary on spreading-vanishing dichotomy and a long time behavior of species. The existence and uniqueness of solutions are verified and the behavior of positive solutions is considered for this system. Moreover, the criteria for spreading-vanishing dichotomy are also derived. The results show that if the length of the initial occupying area is longer than a critical size for the predators or the length of the initial occupying area is shorter than a critical size, but the moving coefficient of free boundary is relatively big, then the spreading of predators always happens under relatively small rate of death for the predator. On the other hand, it is found that if the initial value of free boundary is smaller than a threshold value and the moving coefficient of free boundary is relatively small depending on initial size of predator or the rate of death is relatively big, the predators fail in spreading to new environment.

Keywords: spreading; vanishing; free boundary

1 Introduction

In mathematical ecology, the invasion of immigration for the new species is one of the most important topics. From a viewpoint of mathematical ecology, the various invasion models have been recently put forward and investigated by many ecological mathematicians. For instance, [1–6] proposed reaction-diffusion population models with free boundary to understand the process of the new or invasive population. In [1], the free boundary model is proposed with a logistic diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} - du_{xx} = u(a - bu), & 0 < x < h(t), t > 0, \\ u_x(0,t) = 0, \quad u(x,t) = 0, & x \geq h(t), t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = u_0(x), & 0 \leq x \leq h(t), \end{cases} \tag{1.1}$$

where $x = h(t)$ is the free boundary which will be determined, a, b, d, μ and h_0 are positive constants, and u_0 is a nonnegative initial function. The authors in [1] have considered the uniqueness and existence of global solutions, and derived some interesting results for the dynamics of solution. The vanishing-spreading dichotomy of the population is one of the most remarkable and important results, that is, the solution (u,h) fulfills: $h(t) \to \infty$, $u(x,t) \to a/b$ as $t \to \infty$ or $h(t) \to h_\infty \le \frac{\pi}{2}\sqrt{d/a}$, $u(x,t) \to 0$ as $t \to \infty$.

The following phenomena often happen in the real world. In order to control or kill the pest population, one can put some natural enemies (predators) into a certain area (a bounded region) by employing the biological method because this kind of preys (pest population) first gets into a bounded area (initial habitat) at the initial state and develops very quickly. In this initial habitat (a bounded area), there is some kind of pest population (prey) and another kind of population (predator, the new or invasive species) enters this region to predate at some time (initial time).

During the process of the predators being put into a new habitat, the predators have a tendency to move from the boundary to their new habitat, *i.e.*, they will get into a new bounded area along the free boundary (which is an unknown curve) to predate as time increases. It is reasonable to suppose that the predator invades a new habitat at a rate which is proportional to the gradients of the predators there. Such kind of free boundary conditions has been already introduced in [7–11]. For the more ecological backgrounds of free boundary conditions, one can also refer to [12].

Recently, in [3], the authors have considered the following double free boundary predator-prey problem $\mathbb{R}^1$:

$$\begin{cases} \frac{\partial u}{\partial t} - u_{xx} = u(1-u+av), & g(t) < x < h(t), t > 0, \\ \frac{\partial v}{\partial t} - Dv_{xx} = v(b-v-cu), & x \in \mathbb{R}, t > 0, \\ u = 0, \quad g'(t) = -\mu u_x, & t > 0, x = g(t), \\ u = 0, \quad h'(t) = -\mu u_x, & t > 0, x = h(t), \\ g(0) = -h_0, \quad h(0) = h_0, \\ u(0,x) = u_0(x), \quad x \in [-h_0, h_0], & v(0,x) = v_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\mathbb{R} = (-\infty, \infty)$, $x = h(t)$ and $x = g(t)$ stand for the right and left moving boundaries, respectively, a, b, c, D, h_0 and μ are positive constants. The existence and uniqueness of global solutions of (1.2) and a spreading-vanishing dichotomy have been established. Moreover, the criteria for spreading and vanishing have been obtained in this paper, that is, a spreading critical size $h_0 = \frac{\pi}{2}\sqrt{1/(1+ab)}$ has been derived. Wang in [4] has examined three prey-predator models with free boundary in $\mathbb{R}^1$: DFB, NFB, and TFB. The spreading-vanishing dichotomy, criteria governing spreading-vanishing, and the long time behavior of solution have been provided. For more detailed results, one can refer to [3, 4]. In higher dimension space, Zhao and Wang [13] have considered a Lotka-Volterra competition system incorporating two free boundary with sign-changing coefficients, derived some sufficient conditions for species spreading success and spreading failure, and derived the long time behavior of solutions.

In the process of spreading for the predators, some of them die of starvation, cold and illness. We want to understand how the rate of death impacts on spreading. The behavior of predating always changes by the change of the size of preys and many ecologists observe that the ratio-dependent functional response is more reasonable to describe the process of predating for some predators. Based on these facts, we consider the following ratio-dependent reaction-diffusion predator-prey system with free boundary including a death term:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 u_{xx} = -au + \frac{ecuv}{u+v}, & 0 < x < h(t), t > 0, \\ \frac{\partial v}{\partial t} - d_2 v_{xx} = (1-v)v - \frac{cuv}{u+v}, & x > 0, t > 0, \\ u_x(0,t) = v_x(0,t) = 0, & t > 0, \\ u(x,t) = 0, & x \geq h(t), t > 0, \\ h'(t) = -\mu u_x(h(t),t), \quad h(0) = h_0, & t > 0, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq h_0, & v(x,0) = v_0(x), \quad x \geq 0, \end{cases} \tag{1.3}$$

where $x = h(t)$ represents the moving boundary to be determined; u expresses the population density of the predator species while v stands for the population density of the prey species. h_0, a, b, c, d_i $(i = 1,2)$ and e are positive constants. For what these coefficients stand for the detailed meaning, one can refer to [14]. The initial functions $u_0(x)$ and $v_0(x)$ correspondingly satisfy

$$\begin{cases} u_0 \in C^2([0,h_0]), \quad v_0 \in C^2([0,\infty)), \quad v_0 > 0, \quad x \in [0,\infty), \\ u_0(x) = 0, \quad x \in [h_0,\infty) \quad \text{and} \quad u_0(x) > 0, \quad x \in [0,h_0). \end{cases} \tag{1.4}$$

In this article, we shall show that system (1.3) admits a unique solution and a spreading-vanishing dichotomy holds for this system, namely, as $t \to \infty$, either

(i) the predator $u(x,t)$ spreads successfully to a new environment in the sense that $h(t) \to \infty$,

or

(ii) the predator $u(x,t)$ fails in establishing and vanishes eventually, *i.e.*, $h_\infty < \infty$, $\|u(x,t)\|_{C[0,h(t)]} \to 0$, and $v(x,t) \to 1$.

The criteria for spreading and vanishing are obtained as follows. If the length of the initial occupying area is longer than a critical size for the predators or the length of the initial occupying area is shorter than a critical size, but the moving coefficient of free boundary is relatively big, then the spreading of predators always happens under relatively small rate of death for the predator. On the other hand, if h_0 is smaller than a threshold value and μ is relatively small depending on the initial size of predator or the rate of death is relatively big, the vanishing of predator happens.

Compared with [15], this work mainly has the following differences: (1) it is proved that if the rate of death for the predator is relatively big, then the vanishing of predator happens (Theorem 3.1); (2) new comparison principle is established and then it is used to investigate the criteria for spreading and vanishing; (3) one initial occupying critical size $\tilde{h}_0$ is found (Theorem 3.3) and this value describes that if $h_0 > \tilde{h}_0$, spreading always happens regardless of μ and the initial value (u_0, v_0); (4) when $h_0 \leq \frac{1}{4}\sqrt{\frac{d_1}{ec-a}}$, one critical value $\mu_0 = \frac{d_1}{8M}$ $(M = \frac{4}{3}\|u_0\|_\infty)$ is found and specifically expressed (in [15], the existence of this value is proved, but not expressed specifically), and it shows that if $\mu < \mu_0$, then spreading fails.

This paper is organized into four sections. In the next section, the unique existence of solutions for system (1.3) is established. In Section 3, the spreading-vanishing dichotomy is investigated. In the final section, we make some brief comments and draw conclusions.

2 Existence of solution

Theorem 2.1 *Assume that u_0 and v_0 satisfies* (1.4) *for some $h_0 > 0$. Then, for $0 \leq t < T$ and any $\alpha \in (0,1)$, there exists a unique solution,*

$$(u,v,h) \in C^{1+\alpha,(1+\alpha)/2}(D_T) \times C^{1+\alpha,(1+\alpha)/2}\big(D_T^\infty\big) \times C^{1+\alpha/2}\big([0,T]\big)$$

for system (1.3), *furthermore,*

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(D_T)} + \|v\|_{C^{1+\alpha,(1+\alpha)/2}(D_T^\infty)} + \|h\|_{C^{1+\alpha/2}([0,T])} \leq C,$$

where $D_T = \{(x,t) \in \mathbb{R}^2 : x \in [0,h(t)], t \in [0,T)\}$, $D_T^\infty = \{(x,t) \in \mathbb{R}^2 : x \in [0,\infty), t \in [0,T)\}$, c and T only depend on h_0, α, $\|u_0\|_{C^2[0,h_0]}$ and $\|v_0\|_{C^2([0,\infty))}$.

Proof As in [16], it needs to straighten the free boundary. Assume that $\zeta(s)$ is a function in $C^3[0,\infty)$ fulfilling

$$\begin{aligned} &\zeta(s) = 1 \quad \text{if } |s-h_0| < \frac{h_0}{8}, \qquad \zeta(s) = 0 \quad \text{if } |s-h_0| > \frac{h_0}{2}, \\ &\zeta(s) = 1 \quad \text{if } \big|\zeta'(s)\big| < \frac{5}{h_0} \text{ for all } s. \end{aligned}$$

Let us introduce a transformation

$$(x,t) \to (y,t), \quad \text{where } y = x + \zeta(x)\big(h(t)-h_0\big), x \in \mathbb{R}^1,$$

which yields the transformation

$$(s,t) \to (x,t) \quad \text{with } x = s + \zeta(s)\big(h(t)-h_0\big), 0 \leq s < \infty.$$

For $t \geq 0$, if

$$\big|h(t)-h_0\big| \leq \frac{h_0}{8},$$

the above transformation $x \to y$ is a diffeomorphism from $\mathbb{R}^1$ onto $\mathbb{R}^1$ and the induced transformation $s \to x$ is also a diffeomorphism from $[0,\infty)$ onto $[0,\infty)$. Furthermore, it transforms the free boundary $x = h(t)$ into the line $s = h_0$. Straightforward computations show that

$$\begin{aligned} &\frac{\partial s}{\partial x} = \frac{1}{1+\zeta'(s)(h(t)-h_0)} := \sqrt{A\big(h(t),s\big)}, \\ &\frac{\partial^2 s}{\partial x^2} = -\frac{\zeta''(s)(h(t)-h_0)}{[1+\zeta'(s)(h(t)-h_0)]^3} := B\big(h(t),s\big), \\ &-\frac{1}{h'(t)}\frac{\partial s}{\partial t} = \frac{\zeta(s)}{1+\zeta'(s)(h(t)-h_0)} := C\big(h(t),s\big). \end{aligned} \tag{2.1}$$

Setting

$$\begin{aligned} &u(x,t) = u\big(s+\zeta(s)\big(h(t)-h_0\big),t\big) := \vartheta(s,t), \\ &v(x,t) = v\big(s+\zeta(s)\big(h(t)-h_0\big),t\big) := \varrho(s,t), \end{aligned}$$

then the free boundary system (1.3) can be rewritten as follows:

$$\begin{cases} \vartheta_t - Ad_1\vartheta_{ss} - (Bd_1 + h'C)\vartheta_s = -a\vartheta + \frac{ec\vartheta\varrho}{\vartheta+\varrho}, & t > 0, 0 < s < h_0, \\ \varrho_t - Ad_2\varrho_{ss} - (Bd_2 + h'C)\varrho_s = (1-\varrho)\varrho - \frac{c\vartheta\varrho}{\vartheta+\varrho}, & 0 < s, t > 0, \\ \vartheta_s(0,t) = \varrho_s(0,t) = 0, \quad t > 0, \qquad \vartheta(s,t) = 0, & s \geq h_0, t > 0, \\ h'(t) = -\mu\vartheta_s(h_0,t), \qquad h(0) = h_0, & t > 0, \\ \vartheta(s,0) = u_0(s), \qquad \varrho(s,0) = v_0(s), & 0 \leq s \leq h_0, \end{cases} \tag{2.2}$$

where $A = A(h(t),s)$, $B = B(h(t),s)$, $C = C(h(t),s)$.

The rest of the proof is similar to that of Theorem 2.1 in [1], which follows from the contraction mapping theorem together with standard L^p theory and Sobolev imbedding, so we omit it here. □

To show that the local solution can being extended to all $t > 0$, the following estimate will be employed.

Lemma 2.1 *Assume that* (u,v) *is a bounded solution of* (1.3) *for* $t \in (0,T_0)$ *and* $T_0 \in (0,+\infty]$. *Then there exist positive constants* C_1 *and* C_2 *which are independent of* T_0 *such that*

$$0 < u(x,t) \leq C_1 \quad \textit{for } 0 \leq x < h(t), t \in (0,T_0),$$

$$0 < v(x,t) \leq C_2 \quad \textit{for } 0 \leq x < +\infty, t \in (0,T_0).$$

Proof Let (u,v) be solution of (1.3), then it follows from the strong maximum principle that $u(x,t) > 0$ in $[0,h(t)) \times [0,T_0]$ and $v(x,t) > 0$ in $[0,\infty) \times [0,T_0]$. In addition, by using the maximum principle, we find that there exist positive constants C_1 and C_2 such that $u(x,t) \leq C_1$ in $[0,h(t)) \times [0,T_0]$ and $v(x,t) \leq C_2$ in $[0,\infty) \times [0,T_0]$. □

Lemma 2.2 *Assume that* (u,v) *is a defined solution of* (1.3) *for* $t \in (0,T_0)$ *and* $T_0 \in (0,+\infty]$. *Then there exists a positive constant* C_3 *which is independent of* T_0 *such that*

$$0 < h'(t) \leq C_3 \quad \textit{for } t \in (0,T_0).$$

Proof By using Hopf Lemma to the equation of u, we immediately obtain

$$u(x,t) > 0, \qquad u_x\bigl(h(t),t\bigr) < 0$$

for $0 < t < T_0$, and $0 \leq x < h(t)$. It follows from the Stefan condition that $h'(t) > 0$ for $t \in (0,T_0)$.

We define

$$\Omega = \Omega_A := \bigl\{(x,t) : 0 < h(t) - A^{-1} < x < h(t), 0 < t < T_0\bigr\}$$

as in [2] and establish an auxiliary function

$$w(x,t) := C_1\bigl[2A\bigl(h(t)-x\bigr) - A^2\bigl(h(t)-x\bigr)^2\bigr].$$

We shall show that A can be chosen so that $w(x,t) \geq u(x,t)$ holds over Ω. It is easy to check that, for $(x,t) \in \Omega$,

$$w_t = 2AC_1h'(t)\big(1 - A\big(h(t) - x\big)\big) \geq 0,$$
$$-w_{xx} = 2A^2C_1, \qquad -au + \frac{ecuv}{u+v} \leq ecC_1.$$

It follows that

$$w_t - d_1 w_{xx} \geq 2d_1A^2C_1 - 2$$

if $A^2 \geq \frac{ec}{2d_1}$. Next, we have

$$w\big(h(t) - A^{-1}, t\big) = C_1 \geq u\big(h(t) - A^{-1}, t\big), \qquad w\big(h(t), t\big) = 0 = u\big(h(t), t\big).$$

Thus, for $0 < t < T_0$ and $x \in [h_0 - A^{-1}, h_0]$, if we can take A such that $u_0(x) \leq w(x,0)$, then, by applying the maximum principle to $w - u$ over Ω, we have $u(x,t) \leq w(x,t)$ for $(x,t) \in \Omega$, which yields

$$u_x\big(h(t), t\big) \geq w_x\big(h(t), t\big) = -2AC_1, \qquad h'(t) = -\mu u_x\big(h(t), t\big) \leq 2\mu AC_1.$$

Therefore, it is necessary to find certain A independent of T_0 such that $u_0(x) \leq w(x,0)$ for $x \in [h_0 - A^{-1}, h_0]$.

By direct calculation, we get

$$w_x(x,0) = -2AC_1\big[1 - A(h_0 - x)\big] \leq -AC_1$$

for $x \in [h_0 - (2A)^{-1}, h_0]$. Then, by choosing

$$A := \max\left\{\frac{4\|u_0\|_{C^1([0,h_0])}}{3C_1}, \sqrt{\frac{ec}{2d_1}}\right\}$$

for $x \in [h_0 - (2A)^{-1}, h_0]$, we have

$$w_x(x,0) \leq -AC_1 \leq -\frac{4}{3}\|u_0\|_{C^1([0,h_0])} \leq u_0'(x).$$

Since $w(h_0, 0) = u_0(h_0) = 0$, the above inequality yields $w(x,0) \geq u_0(x)$. Furthermore, for $x \in [h_0 - A^{-1}, h_0 - (2A)^{-1}]$, one can easily find that

$$w(x,0) \geq \frac{3}{4}C_1, \qquad u_0(x) \leq \|u_0\|_{C^1([0,h_0])}A^{-1} \leq \frac{3}{4}A_1.$$

Therefore, $u_0(x) \leq w(x,0)$ for $x \in [h_0 - A^{-1}, h_0]$. This completes the proof. □

Employing Lemma 2.1 and Lemma 2.2, the local solution of (1.3) can be extended for all $t > 0$ by the regular argument, that is, one can obtain the following results.

Theorem 2.2 *There exists a unique solution of system* (1.3) *for all* $t > 0$.

3 The spreading-vanishing dichotomy

From Lemma 2.2 it follows that $x = h(t)$ is monotonic increasing. Thus, $\lim_{t\to\infty} h(t) = h_\infty \in (0,\infty]$. The following result shows that the predator will not spread successfully in the case $a > ec$.

Theorem 3.1 *Suppose that $a > ec$, then $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$ and $h_\infty < \infty$. Furthermore, $\lim_{t\to\infty} v(r,t) = 1$ holds uniformly in any bounded subset of $[0,\infty)$.*

Proof By the comparison principle, one can get $v(x,t) \le \overline{v}(t)$ for $x \ge 0$ and $t \in (0,\infty)$, where

$$\overline{v}(t) = \frac{e^t\|v_0\|_\infty}{1 + (e^t - 1)\|v_0\|_\infty}$$

is the solution of the ODE problem

$$\frac{d\overline{v}}{dt} = \overline{v}(1 - \overline{v}), \quad t > 0, \qquad \overline{v}(0) = \|v_0\|_\infty.$$

Since $\lim_{t\to\infty} \overline{v}(t) = 1$, it is deduced that $\limsup_{t\to\infty} v(t) \le 1$ holds uniformly in any bounded subset of $[0,\infty)$. From the condition $a > ec$ it follows that there exists a small ε such that $a > \frac{ec(1+\varepsilon)}{u+b(K+\varepsilon)}$. On the other hand, for this ε, there exists T_0 such that $v(x,t) \le 1+\varepsilon$ in $[0,+\infty) \times [T_0,\infty)$. Then $u(x,t)$ satisfies

$$\begin{cases} u_t - d_1u_{xx} \le (-a + \frac{ec(1+\varepsilon)}{u+(1+\varepsilon)})u(x,t), & 0 < x < h(t), t > T_0, \\ u(x,t) = 0, \qquad u_x(0,t) = 0, & x = h(t), t > 0, \\ u(x,T_0) > 0, & 0 \le x \le h(T_0). \end{cases} \tag{3.1}$$

Using comparison again, we find that $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$. This, together with the condition $u(x,t) = 0$ for $t > 0$, $x \ge h(t)$, shows that there exists $T_\varepsilon > 0$ such that $u(x,t) < \varepsilon$ for any given $0 < \varepsilon \ll 1$, $t > T_\varepsilon$ and $x > 0$. Then the function $v(x,t)$ satisfies

$$\begin{cases} v_t - d_2v_{xx} \ge v(1 - v - \frac{\varepsilon c}{\varepsilon+v}), & 0 < r, t > T_\varepsilon, \\ v_x(0,t) = 0, & t > 0, \\ v(x,T_\varepsilon) > 0, & 0 < x. \end{cases} \tag{3.2}$$

By using the comparison principle again, we obtain

$$\liminf_{t\to\infty} v(x,t) \ge \frac{1 - \varepsilon + \sqrt{(\varepsilon-1)^2 + 4\varepsilon(1-c)}}{2}$$

holds uniformly in any bounded subset of $[0,+\infty)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\liminf_{t\to\infty} v(x,t) \ge 1$ uniformly in any bounded subset of $[0,\infty)$.

Hence, $\lim_{t\to\infty} v(x,t) = 1$ holds uniformly in any bounded subset of $[0,\infty)$.

Next, use Lemma 3.2 of [17] to system (3.1), one can obtain $h_\infty < \infty$. This completes the proof. □

The following comparison principle can be used to estimate the solution $(u(x,t), v(x,t))$ and free boundary $x = h(t)$.

Lemma 3.1 (The comparison principle) *Suppose that* $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{v} \in C((0,\infty) \times [0,T]) \cap C^{2,1}((0,\infty) \times (0,T])$, $\overline{u} \in C(\overline{D}_T^*) \cap C^{2,1}(D_T^*)$ *with* $D_T^* = \{(x,t) \in \mathbb{R}^2 : 0 < x < \overline{h}(t), 0 < t \leq T\}$, *and*

$$\begin{cases} \overline{u}_t - d_1\overline{u}_{xx} \geq -a\overline{u} + \frac{ec\overline{u}\overline{v}}{\overline{u}+\overline{v}}, \quad 0 < x < \overline{h}(t), 0 < t \leq T, \\ \overline{v}_t - d_2\overline{v}_{xx} \geq (1-\overline{v})\overline{v}, \quad x > 0, 0 < t \leq T, \\ \overline{u}_x(0,t) = \overline{v}_x(0,t) = 0, \quad 0 < t \leq T, \\ \overline{u}(x,t) = 0, \quad x \geq \overline{h}(t), 0 < t \leq T, \\ \overline{h}'(t) \geq -\mu\overline{u}_x(\overline{h}(t),t), \qquad \overline{h}(0) \geq h_0, \quad 0 < t \leq T, \\ \overline{u}(x,0) \geq u_0(x), \quad 0 \leq x \leq \overline{h}_0, \qquad \overline{v}(x,0) \geq v_0(x), \quad x \geq 0. \end{cases} \tag{3.3}$$

Then the solution $(u(r,t), v(r,t), h(t))$ *of the free boundary problem* (1.3) *fulfills*

$$u(x,t) \leq \overline{u}(x,t) \quad \text{for } x \in \big(0,h(t)\big) \text{ and } t \in [0,T],$$
$$v(x,t) \leq \overline{v}(x,t), \qquad h(t) \leq \overline{h}(t) \quad \text{for } x \in (0,\infty) \text{ and } t \in [0,T].$$

Proof First of all, the application of the comparison principle to the equations of v and $\overline{v}$ yields $\overline{v} > v$ directly. Since the function $\frac{uv}{u+v}$ is increasing in u for $u, v \geq 0$, by employing the comparison principle given by [1] for the single equation to $\overline{u}$ and u, one can get $\overline{u} > u$ directly. The regular arguments and the detailed proofs are omitted. □

It is well known that the principal eigenvalue $\lambda_1(L)$ of the problem

$$\begin{cases} -\phi_{xx} = \lambda\phi, \quad x \in (0,L), \\ \phi = 0, \quad x = 0, x = L \end{cases} \tag{3.4}$$

is a strictly decreasing continuous function and

$$\lim_{L \to 0^+} \lambda_1(L) = \infty, \qquad \lim_{L \to +\infty} \lambda_1(L) = 0.$$

Hence, there exists a unique L^* ($L^* > 0$) such that

$$\lambda_1\big(L^*\big) = \frac{ec}{d_1},$$

$\lambda_1(L) < \frac{ec}{d_1}$ for $L > L^*$ and $\lambda_1(L) > \frac{ec}{d_1}$ for $L < L^*$.

Lemma 3.2 *Suppose that* $h_0 < L^*$. *Then there exists* $\mu_0 > 0$ *which depends on* u_0 *such that the predator fails in spreading if* $\mu \leq \mu_0$.

Proof For $t > 0$ and $x \in (0, \sigma(t))$, define

$$\sigma(t) = h_0\left(1 + \delta - \frac{\delta}{2}e^{-\gamma t}\right), \qquad \omega(x,t) = Me^{-\gamma t}W\big(h_0x/\sigma(t)\big),$$

where M, δ, γ are positive constants to be taken later and $W(x)$ is the first eigenfunction of the problem

$$\begin{cases} -W_{xx} = \lambda_1(h_0)W, \quad x \in (0,h_0), \\ W = 0, \quad x = 0, x = h_0, \end{cases} \tag{3.5}$$

with $W \geq 0$ and $\|W\|_\infty = 1$. From $h_0 < L^*$, it follows that

$$\lambda_1(h_0) > \frac{ec}{d_1}.$$

By (3.5), it is obvious that $W'(0) = 0$. Therefore, it is deduced that $W'(x) < 0$ for $0 < x \leq h_0$.

Set $\tau(t) = 1 + \delta - \frac{\delta}{2}e^{-\gamma t}$ so that $\sigma(t) = h_0\tau(t)$. By direct calculations, it is derived that

$$\begin{aligned}
&\omega_t - d_1\omega_{xx} - \omega(-a + ec) \\
&\quad = Me^{-\gamma t}\bigl[-\gamma W - x\tau^{-2}\tau'(t)W' - d_1\tau^{-2}W'' - W(-a + ec)\bigr] \\
&\quad \geq Me^{-\gamma t}\bigl[-\gamma + d_1\lambda_1(h_0) + a - ec\bigr]W.
\end{aligned} \tag{3.6}$$

Hence, by $\lambda_1(h_0) > ec$, one can take $\gamma < a$ such that, for $t > 0$, $x \in [0, \sigma(t)]$,

$$\omega_t - d_1\omega_{xx} - \omega(ec - a) \geq 0,$$

which indicates

$$\omega_t - d_1\omega_{xx} + \omega\left(a - \frac{ecv}{u + v}\right) \geq 0$$

for $t > 0$, $x \in [0, \sigma(t)]$.

Now, take $M > 0$ large enough such that

$$u_0(x) \leq MW\left(\frac{x}{1 + \delta/2}\right) = \omega(x, 0) \quad \text{for } x \in [0, h_0].$$

Note that

$$\begin{aligned}
&\sigma'(t) = \frac{1}{2}h_0\gamma\delta e^{-\gamma t}, \quad 1 + \frac{\delta}{2} \leq \tau(t) \leq 1 + \delta, \\
&-\mu\omega_x\bigl(t, \sigma(t)\bigr) = \mu Me^{-\gamma t}\frac{h_0}{\sigma(t)}\bigl|W_x(h_0)\bigr| \\
&\qquad \leq \mu Me^{-\gamma t}\frac{|W_x(h_0)|}{1 + \delta/2}.
\end{aligned}$$

Hence, by taking

$$\mu_0 = \frac{\delta(1 + \delta/2)\gamma h_0}{2M|W_x(h_0)|},$$

we find that $\sigma'(t) \geq -\mu\omega_x(t, \sigma(t))$ for any $0 < \mu \leq \mu_0$.

Let $\overline{v}$ be a unique positive solution of

$$\frac{dv}{dt} = v(1 - v), \quad t > 0, \qquad \overline{v}(0) = \|v_0\|_\infty,$$

and thus $(\omega, \overline{v}, \sigma)$ satisfies

$$\begin{cases} \omega_t - d_1\omega_{xx} \geq -a\omega + ec\omega\overline{v}/(\omega + \overline{v}), & 0 < x < \sigma(t), 0 < t \leq T, \\ \overline{v}_t - d_2\overline{v}_{xx} \geq (1 - \overline{v})\overline{v}, & 0 < x, 0 < t \leq T, \\ \omega_x(0,t) = 0, \quad \overline{v}_x = 0, & 0 < t \leq T, \\ \omega(x,t) = 0, & x \geq \sigma(t), 0 < t \leq T, \\ \sigma(0) = (1 + \delta/2)h_0 > h_0, & 0 < t \leq T, \\ \omega(x,0) \geq u_0(x), & 0 \leq x, \\ \overline{v}(0) = \|v_0\|_\infty, & 0 \leq x, \\ \sigma'(t) \geq -\mu\omega_x(t, \sigma(t)), & 0 < t \leq T. \end{cases} \tag{3.7}$$

Hence, by the comparison principle (Lemma 3.1), one can conclude that

$$h(t) \leq \sigma(t), \qquad u(x,t) \leq \omega(x,t) \quad \text{and} \quad v(x,t) \leq \overline{v}(t)$$

for $0 \leq x \leq h(t)$ and $t > 0$. It follows that

$$h_\infty \leq \lim_{t\to\infty} \sigma(t) = h_0(1 + \delta) < \infty.$$

This completes the proof. □

Theorem 3.2 *Suppose that $a < ec$ and $h_0 \leq \frac{1}{4}\sqrt{\frac{d_1}{ec-a}}$. If $\mu \leq \frac{d_1}{8M}$, then $h_\infty \leq 4h_0 < \infty$, where $M = \frac{4}{3}\|u_0\|_\infty$.*

Proof First, we construct a suitable upper solution to system (1.3) by defining

$$\overline{u} = \begin{cases} Me^{-\gamma t}H(x/\overline{h}(t)), & 0 \leq x \leq \overline{h}(t), \\ 0, & x > h(t), \end{cases} \qquad \overline{v}(x,t) = M_1,$$

where $M_1 = \max\{\|v_0\|_\infty, 1\}$, and

$$\overline{h}(t) = 2h_0\bigl(2 - e^{-\gamma t}\bigr), \qquad t \geq 0, \qquad H(y) = 1 - y^2, \quad 0 \leq x \leq 1,$$

where M_1, γ and M are positive constants to be taken later.

Straightforward computation yields

$$\begin{aligned} &\overline{u}_t - d_1\overline{u}_{xx} + a\overline{u} - \frac{ec\overline{uv}}{\overline{u} + \overline{v}} \\ &\quad = Me^{-\gamma t}\bigl[-\gamma H - x\overline{h}^{-2}\overline{h}'H' - d_1\overline{h}^{-2}H'' + (a - ec)\bigr] \\ &\quad \geq Me^{-\gamma t}\left(-\gamma + a - ec + \frac{d_1}{8h_0^2}\right), \\ &\overline{v}_t - d_2\overline{v}_{xx} - \overline{v}(1 - \overline{v}) \geq 0 \end{aligned} \tag{3.8}$$

for all $0 < x < \overline{h}(t)$ and $t > 0$. In addition, one can easily check that $\overline{h}'(t) = 2h_0\gamma e^{-\gamma t}$ and $-\mu\overline{u}_x(h(t),t) = 2M\mu\overline{h}^{-1}e^{-\gamma t}$. Furthermore, we note that $\overline{u}(x,0) = M(1 - x^2/4h_0^2) \geq \frac{3}{4}M$, $\overline{v}(x,0) \geq v_0(x)$ for $x \in (0,h_0]$. Since $\overline{h}(t) \leq 4h_0$, we choose $M = \frac{4}{3}\|u_0\|_\infty$ and $\gamma = \frac{d_1}{16h_0^2}$,

$\mu \le \frac{d_1}{8M}$, where $h_0 \le \frac{1}{4}\sqrt{\frac{d_1}{ec-a}}$. Then we find that (3.3) holds. Hence, by Lemma 3.1, we find that $h(t) \le \overline{h}(t)$ for $t > 0$ and $h_\infty \le \lim_{t\to\infty} \overline{h}(t) = 4h_0 < \infty$. This completes the proof. □

Lemma 3.3 *Suppose that* $h_\infty < \infty$. *Then* $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)))} = 0$ *and* $\lim_{t\to\infty} v(x,t) = 1$ *hold uniformly in any bounded subset of* $[0,\infty)$.

Proof Suppose for contradiction that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = \sigma > 0.$$

Then, for all $m \in \mathbb{N}$, there exists a sequence (x_m, t_m) in $[0,h(t)) \times (0,\infty)$ such that $u(x_m,t_m) \ge \sigma/2$ and $t_m \to \infty$ as $m \to \infty$. From $0 \le x_m < h(t) < h_\infty < \infty$ it follows that a subsequence of $\{x_n\}$ converges to $x_0 \in [0,h_\infty)$. Without loss of generality, denote this subsequence still by $\{x_n\}$.

Define $u_m(x,t) = u(x,t_m+t)$ and $v_m(x,t) = v(x,t_m+t)$ for $(x,t) \in (0,h(t_m+t)) \times (-t_m,\infty)$. By the parabolic regularity, we find that there exists a subsequence $\{(u_{m_i},v_{m_i})\}$ of $\{(u_m,v_m)\}$ such that $(u_{m_i},v_{m_i}) \to (\tilde{u},\tilde{v})$ as $i \to \infty$ and $(\tilde{u},\tilde{v})$ satisfies

$$\begin{cases} \tilde{u}_t - d_1\tilde{u}_{xx} = -a\tilde{u} + \frac{ec\tilde{u}\tilde{v}}{\tilde{u}+\tilde{v}}, & (x,t) \in (0,h_\infty) \times (-\infty,+\infty), \\ \tilde{v}_t - \tilde{v}_{xx} = \tilde{v}(1-\tilde{v}) - \frac{c\tilde{u}\tilde{v}}{\tilde{u}+\tilde{v}}, & (x,t) \in (0,h_\infty) \times (-\infty,+\infty). \end{cases} \tag{3.9}$$

From $\tilde{u}(x_0,0) \ge \sigma/2$, it follows that $\tilde{u} > 0$ in $(0,h_\infty) \times (-\infty,+\infty)$. Since $-a + \frac{ec\tilde{v}}{\tilde{u}+\tilde{v}}$ is bounded by $R := a + ec$, by using Hopf lemma to the equation $\tilde{u}_t - d_1\tilde{u}_{xx} \ge -R\tilde{u}$ at the point $(h_\infty,0)$, one can find that $\tilde{u}_x(h_\infty,0) \le -\delta_0$ for some $\delta_0 > 0$.

As in [18], we define

$$s = \frac{h_0 x}{h(t)}, \qquad \vartheta(s,t) = u(x,t), \qquad \varrho(s,t) = v(x,t),$$

then straightforward calculations yield

$$u_t = \vartheta_t - \frac{h'(t)}{h(t)}s\vartheta_s, \qquad u_x = \frac{h_0}{h(t)}\vartheta_s, \qquad u_{xx} = \frac{h_0^2}{h^2(t)}\vartheta_{ss}.$$

Therefore, $\vartheta(s,t)$ fulfils

$$\begin{cases} \vartheta_t - d_1\frac{h_0^2}{h^2(t)}\vartheta_{ss} - \frac{h'(t)}{h(t)}s\vartheta_s = \vartheta(-a + \frac{ec\varrho}{\vartheta+\varrho}), & 0 < s < h_0, t > 0, \\ \vartheta_s(0,t) = \vartheta(h_0,t) = 0, & t > 0, \\ \vartheta(s,0) = u_0(s) \ge 0, & 0 \le s \le h_0. \end{cases} \tag{3.10}$$

The free boundary $x = h(t)$ is straightened as the fixed line $s = h_0$. By Proposition A in [19], there exists a positive constant K_0 such that

$$\|\vartheta\|_{C^{\frac{1+\alpha}{2},1+\alpha}([1,\infty)\times[0,h(t)])} < K_0,$$

which shows that there exists a constant K such that

$$\|u(\cdot,t)\|_{C^1([0,h(t)])} < K, \quad \forall t \ge 1.$$

By $h'(t) = -\mu u_x(h(t), t)$, $0 < h'(t) < M_3$ and $\|\vartheta_s(s,t)\|_{C^{\frac{\alpha}{2}}([1,\infty))} < K_0$, one can obtain

$$\|h'\|_{C^{\frac{\alpha}{2}}([1,\infty))} < M_4, \tag{3.11}$$

where M_4 depends on K_0 and M_3. Then, by using Proposition 3.2 of [3], one can directly obtain $\lim_{t\to\infty} \|u\|_{C([0,h(t)])} = 0$. This completes the proof. □

Remark 3.1 If the predator fails in spreading, then it will be extinct finally.

Theorem 3.3 *Suppose that $a < ec$, then $h_\infty = \infty$ if $h_0 > \tilde{h}_0$, where $\lambda_1(\tilde{h}_0) = \frac{1}{d_1}(ec - a) > 0$.*

Proof Assume, for contradiction, that $h_\infty < \infty$. Then it follows from Lemma 3.3 that

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0 \quad \text{and} \quad \lim_{t\to\infty} v(1,t) = 1,$$

uniformly in any bounded subset $[0, h_0]$. Hence, for any $\varepsilon > 0$, there exists $\widetilde{T} > 0$ such that $u(x,t) \le \varepsilon$ and $v(x,t) \ge 1 - \varepsilon$ for $t \ge \widetilde{T}$, $x \in [0, h(t)]$.

Note that $u(x,t)$ satisfies

$$\begin{cases} u_t - d_1 u_{xx} \ge u[-a + \frac{ec(1-\varepsilon)}{\varepsilon+(1-\varepsilon)}], & 0 < x < h_0, t > \widetilde{T}, \\ u_x(0,t) = 0, \qquad u(h_0,t) \ge 0, & t > \widetilde{T}, \\ u(x,\widetilde{T}) > 0, \quad 0 \le x < h_0. \end{cases} \tag{3.12}$$

Let $\underline{u}(x,t)$ satisfy the following problem:

$$\begin{cases} u_t - d_1 u_{xx} = u[-a + ec(1-\varepsilon)], & 0 < x < h_0, t > \widetilde{T}, \\ u_x(0,t) = 0, \qquad u(h_0,t) = 0, & t > \widetilde{T}, \\ u(x,\widetilde{T}) > 0, \quad 0 \le x < h_0. \end{cases} \tag{3.13}$$

Then $\underline{u}(x,t)$ is a lower solution of $u(x,t)$. Since $h_0 > \tilde{h}_0$, one can choose ε small enough such that

$$d_1\lambda_1(h_0) < -a + ec(1-\varepsilon).$$

By the condition $a < ec$, it follows from the well-known result that $\underline{u}$ is unbounded in $(0, h_0) \times [T^*, \infty)$, which contradicts that $\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} = 0$. This completes the proof. □

Remark 3.2 Since $\lambda_1(\tilde{h}_0) = \frac{1}{d_1}(ec - a) < \frac{ec}{d_1} = \lambda_1(L^*)$, one can find that $\tilde{h}_0 > L^*$, which is different from the result of [1].

Theorem 3.4 *Assume that $a < ec$ and $h_0 \le \frac{1}{4}\sqrt{d_1/(ec - a)}$. Then there exists μ_1 depending on $u_0(x)$ and $v_0(x)$ such that $h_\infty = \infty$ if $\mu > \mu_1$ for system* (1.3).

Proof The proof is similar to that of Lemma 3.2 in [20]. For convenience of the reader, it is included here. It follows from (1.3) that there exists a constant $\sigma_* > 0$ such that

$$u_t - d_1 u_{xx} = -au + \frac{ecuv}{u + v} \ge -\sigma_* u, \quad 0 < x < h(t).$$

Construct and consider the following auxiliary free boundary system:

$$\begin{cases} z_t - d_1 z_{xx} = -\sigma_* z, & 0 < x < r(t), t > 0, \\ z_x(0,t) = 0, & t > 0, \\ z(x,t) = 0, \quad r(t) = -\mu z_x, & t > 0, x = r(t), \\ z(x,0) = u_0(x), & 0 \le x < h_0, \\ r(0) = h_0. \end{cases} \tag{3.14}$$

By using the comparison principle, one can find that $z(x,t) \le u(x,t)$ and $r(t) \le h(t)$ for all $t \ge 0$ and $0 \le x \le r(t)$. By using similar argument to the proof of Lemma 3.2 in [20], one can find that there exists a constant $\mu_1 > 0$ such that

$$r(2) \ge \pi \sqrt{d_1/(ec-a)}$$

for all $\mu \ge \mu_1$. Therefore, it is derived that

$$h_\infty = \lim_{t\to\infty} h(t) \ge r(2) \ge \pi \sqrt{d_1/(ec-a)} > 4h_0, \quad \forall \mu \ge \mu_1.$$

This, together with Theorem 3.2, yields the desired result. □

Using a similar argument to the proof of Theorem 3.2 in [15], one can prove the following theorem, which shows that the predator establishes itself successfully in the new environment in the sense that $h_\infty = \infty$ if the rate of death for the predator is relatively small. Moreover, in this case, both predator and prey can coexist for a long time.

Theorem 3.5 *Assume that $a < (1-c)ec$ and $h_\infty = \infty$. Then the solution (u,v,h) of system* (1.3) *satisfies*

$$\lim_{t\to} u(x,t) = \frac{ec-a}{a}\left(1 - c + \frac{a}{e}\right),$$
$$\lim_{t\to} v(x,t) = 1 - c + \frac{a}{e},$$

uniformly in any compact subset of $[0,\infty)$.

The proof of the theorem is similar to that of Theorem 3.2 in [15], so it is omitted.

Theorem 3.6 *Assume that $ec > a$ and $h_0 < \frac{1}{4}\sqrt{d_1/(ec-a)}$. There exists $\mu^* > \mu_* > 0$ which depends on $u_0(x)$ and $v_0(x)$, such that $h_\infty < 4h_0$ if $\mu < \mu_*$ and $h_\infty = \infty$ if $\mu > \mu^*$ for system* (1.3).

The proof of Theorem 3.6 is essentially the same as that of Theorem 5.2 in [4] and thus is omitted.

4 Comments and conclusions

In this article, we have investigated a ratio-dependent diffusion predator-prey system with the free boundary $x = h(t)$, which describes the process of movement for the predator species.

For the successful spreading of predator to a new environment for this model, only one result is derived, that is, the predator $u(x,t)$ spreads successfully to a new environment in the sense that $h(t) \to \infty$ if $a < ec$ and $h_0 > \tilde{h}_0$, where $\lambda_1(\tilde{h}_0) = \frac{1}{d_1}(ec - a) > 0$.

Assume one of the following three cases holds: (i) $a < ec$, $h_0 \le \frac{1}{4}\sqrt{\frac{d_1}{ec-a}}$ and $\mu \le \frac{d_1}{8M}$, then $h_\infty < \infty$, where $M = \frac{4}{3}\|u_0\|_\infty$; (ii) $a > ec$; (iii) $h_0 < L^*$ and $\mu \le \mu_0$, where $\mu_0 > 0$ depending on u_0. Then the predator $u(x,t)$ fails in establishing itself and vanishes finally, *i.e.*, $h_\infty < \infty$, $\|u(x,t)\|_{C[0,h(t)]} \to 0$ and $v(x,t) \to 1$.

Therefore, the criteria for spreading and vanishing are as follows. If the death rate of predator is relatively small and the length of the initial occupying area is longer than a critical size $\tilde{h}_0$, then the spreading of predator always happens. For vanishing of the predator, there are three criteria: (i) the rate of death is bigger than a critical value ec; (ii) the length of initial occupying area h_0 is shorter than a threshold value L^* and μ is smaller than the critical value μ_0, depending on u_0; (iii) the length of the initial occupying area h_0 is shorter than $\frac{1}{4}\sqrt{\frac{d_1}{ec-a}}$ and μ is smaller than $\frac{d_1}{8M}$, depending on u_0.

From the above results of the dichotomy, it follows that in order to control the prey population (pest species) one should at least put predator population (natural enemies) into the initial habitat at the initial state in one of four ways: (i) decrease the death rate of predator during the process of putting; (ii) extend the range of predator's targets; (iii) accelerate putting predators; (iv) choose the natural enemies which have a strong ability for predating.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author thanks the anonymous referee very much for the valuable comments and suggestions to improve the contents of this article. This work was supported by the National Natural Science Foundation of China (11461023) and the research funds of PhD for Honghe University (14bs19). The author also thanks Prof. Wang Mingxin very much for providing reference [15] which was important for work in the revision stage.

Received: 21 June 2016 Accepted: 20 October 2016 Published online: 03 November 2016

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